4.2 Null space, Column space, and linear transformations

Key idea: Every $m \times n$ matrix A has two fundamental subspaces associated to it. The null space in $\mathbb{R}^n$ and the column space in $\mathbb{R}^m$. We define these and connect them to the fundamental subspaces of a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ (the kernel in $\mathbb{R}^n$ and the range in $\mathbb{R}^m$).

Subspaces of $\mathbb{R}^n$ arise frequently in two specific ways:

1) as the set of solutions to some system of homogeneous linear equations

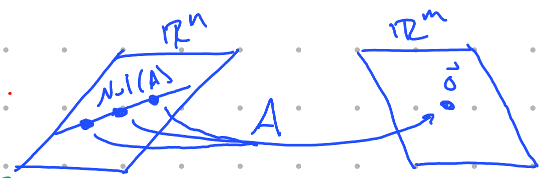
or 2) as the span of some collection of specified vectors.

We will see that any matrix A $(m \times n)$ defines a subspace of $\mathbb{R}^n$ in the first manner, and a subspace of $\mathbb{R}^m$ in the second manner.

Def: The null space of a $(m\times n)$ matrix A is the set of all $\vec{x} \in \mathbb{R}^n$ s.t. $A\vec{x} = \vec{0}$.

We write

$$\mathrm{Nul}(A) = \{\vec{x} \text{ in } \mathbb{R}^n : A\vec{x} = \vec{0}\}.$$



Ex $\vec{u} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$ is in the null space of $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$.

To see this note $A\vec{u} = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ -25 \end{bmatrix} + \begin{bmatrix} -9 \\ 27 \end{bmatrix} + \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{0}$

Notice also $\mathrm{Nul}(A)$ is the solution set of $A\vec{x} = \vec{0}$ or as desired.

$$x_1 - 3x_2 - 2x_3 = 0$$
$$-5x_1 + 9x_2 + x_3 = 0$$

Let's verify that for any A $(m \times n)$, $\mathrm{Nul}(A)$ is indeed a subspace of $\mathbb{R}^n$.

1) $\vec{0}$ in $\mathrm{Nul}(A)$: $A\vec{0} = \vec{0}$. ✓

2) $\vec{u}, \vec{v}$ in $\mathrm{Nul}(A)$: $A\vec{u} = \vec{0}$, $A\vec{v} = \vec{0}$ $\Rightarrow$ $A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v} = \vec{0} + \vec{0} = \vec{0}$. ✓
$\Rightarrow \vec{u} + \vec{v}$ in $\mathrm{Nul}(A)$

3) $\vec{u}$ in $\mathrm{Nul}(A)$: $A\vec{u} = \vec{0}$ $\Rightarrow$ $A(c\vec{u}) = cA\vec{u} = c\vec{0} = \vec{0}$. ✓
$\Rightarrow c\vec{u}$ in $\mathrm{Nul}(A)$

Ex| Describe the null space of $A=\begin{bmatrix}-3&6&-1&1&-7\\1&-2&2&3&-1\\2&-4&5&8&-4\end{bmatrix}$ by finding a spanning set.

Notice a parametric description of the solution set of $A\vec{x}=\vec{0}$ describes every element of Nul(A) as a linear combination, i.e., it shows an element in the span of some specific vectors.

$$A=\begin{bmatrix}-3&6&-1&1&-7\\1&-2&2&3&-1\\2&-4&5&8&-4\end{bmatrix}\sim\begin{bmatrix}1&-2&0&-1&3\\0&0&1&2&-2\\0&0&0&0&0\end{bmatrix} \quad \text{so } [A\ \vec{0}]\sim\begin{bmatrix}1&-2&0&-1&3&0\\0&0&1&2&-2&0\\0&0&0&0&0&0\end{bmatrix}$$

x_1 x_2 x_3 x_4 x_5

$\nwarrow$ free's

which implies $A\vec{x}=\vec{0}$ if $\vec{x}=\begin{bmatrix}2x_2+x_4-3x_5\\x_2\\-2x_4+2x_5\\x_4\\x_5\end{bmatrix}=x_2\begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix}+x_4\begin{bmatrix}1\\0\\-2\\1\\0\end{bmatrix}+x_5\begin{bmatrix}-3\\0\\2\\0\\1\end{bmatrix}$.

That is, if $\vec{x}$ in $\text{Span}\left\{\begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix},\begin{bmatrix}1\\0\\-2\\1\\0\end{bmatrix},\begin{bmatrix}-3\\0\\2\\0\\1\end{bmatrix}\right\} \implies \text{Nul}(A)=\text{Span}\left\{\begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix},\begin{bmatrix}1\\0\\-2\\1\\0\end{bmatrix},\begin{bmatrix}-3\\0\\2\\0\\1\end{bmatrix}\right\}$.

As x_2, x_4, x_5 are free, these vectors are lin. indep. So Nul(A) is a 3D subspace of $\mathbb{R}^5$

Now, Nul(A) is the subspace of $\mathbb{R}^n$ associated to an $m\times n$ matrix, we describe next the associated subspace of $\mathbb{R}^m$.

Def: The **column space** of an $m\times n$ matrix $A=[\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_n]$ is set of all linear combinations of the columns of A. We write

$$\text{Col}(A)=\text{Span}\{\vec{a}_1,\vec{a}_2,\dots,\vec{a}_n\}$$
$$\|$$
$$\{\vec{b} \text{ in } \mathbb{R}^m : A\vec{x}=\vec{b} \text{ for some } \vec{x} \text{ in } \mathbb{R}^n\}$$

Notice Col(A) is a subspace due to it being a spanning set.

Ex| Let $A=\begin{bmatrix}2&4&-2&1\\-2&-5&7&3\\3&7&-8&6\end{bmatrix}$. A is 3×4 so Col(A) is in $\mathbb{R}^3$ and Nul(A) is in $\mathbb{R}^4$. This is b/c A maps $\mathbb{R}^4$ to $\mathbb{R}^3$.

Is $\vec{v}=\begin{bmatrix}3\\1\\3\end{bmatrix}$ in Col(A)? $\rightarrow$ equivalently, is $\vec{v}$ in $\text{span}\left\{\begin{bmatrix}2\\-2\\3\end{bmatrix},\begin{bmatrix}4\\-5\\7\end{bmatrix},\begin{bmatrix}-2\\7\\8\end{bmatrix},\begin{bmatrix}1\\3\\6\end{bmatrix}\right\}$?

Recall this is equivalent to finding $\vec{x}$ s.t. $A\vec{x}=\vec{v}$.

So row reduce $[A\ \vec{v}]=\begin{bmatrix}2&4&-2&1&3\\-2&-5&7&3&1\\3&7&-8&6&3\end{bmatrix}\sim\begin{bmatrix}2&4&-2&1&3\\0&1&-5&-4&-2\\0&0&0&17&1\end{bmatrix}$.

So, we see that $A\vec{x}=\vec{v}$ is consistent, i.e. - - - $\vec{v}$ in $\text{span}\{\vec{a}_1,\dots,\vec{a}_5\}$ so yes, $\vec{v}$ in Col(A).

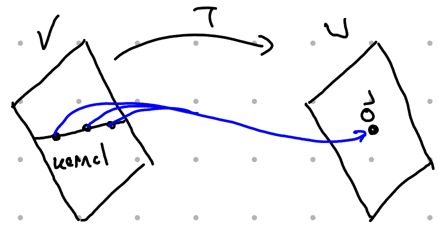
Could $\vec{v}$ be in Nul(A)? No, Nul(A) is in $\mathbb{R}^5$ but $\vec{v}$ is in $\mathbb{R}^3$.

Recall matrices represent linear transformations so these subspaces have natural interpretations in that setting.

Def: A linear transformation T from a vector space V to another vector space W is a rule which assigns to each $\vec{x}$ in V a vector $T(\vec{x})$ in W such that:

1) $T(\vec{u}+\vec{v}) = T(\vec{u})+T(\vec{v})$ and 2) $T(c\vec{u}) = cT(\vec{u})$

for all $\vec{u},\vec{v}$ in V and scalars c.



We then define analogous spaces:

The kernel of T are all $\vec{x}$ in V s.t. $T(\vec{x}) = \vec{0}$.

The range of T are all $\vec{y}$ in W s.t. $T(\vec{x}) = \vec{y}$ for some $\vec{x}$ in V.

Thus we have the analogy: if $\underset{m\times n}{A}$ and $T: \mathbb{R}^n \to \mathbb{R}^m$ defined by $x \mapsto A\vec{x}$

then

$$\mathrm{Nul}(A) = \mathrm{Kernel}(T)$$
$$\mathrm{Col}(A) = \mathrm{Range}(T)$$

Next example: Let V be the space of all differentiable functions. Then $D: V \to V$ defined by $D(f) = f'$ is a linear transformation:

$$D(f+g) = (f+g)' = f'+g' = D(f)+D(g)$$

$$D(cf) = (cf)' = cf' = cD(f).$$

What then is the kernel of D and the range of D?

To finish, we consider the handout:

"Comparing and contrasting $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$"

Contrast Between Nul *A* and Col *A* for an *m* x *n* Matrix *A*

Nul A	Col A
1. Nul A is a subspace of $\mathbb{R}^n$.	1. Col A is a subspace of $\mathbb{R}^m$.
2. Nul A is implicitly defined; that is, you are given only a condition ($A\mathbf{x} = \mathbf{0}$) that vectors in Nul A must satisfy.	2. Col A is explicitly defined; that is, you are told how to build vectors in Col A.
3. It takes time to find vectors in Nul A. Row operations on $[\,A \quad \mathbf{0}\,]$ are required.	3. It is easy to find vectors in Col A. The columns of A are displayed; others are formed from them.
4. There is no obvious relation between Nul A and the entries in A.	4. There is an obvious relation between Col A and the entries in A, since each column of A is in Col A.
5. A typical vector $\mathbf{v}$ in Nul A has the property that $A\mathbf{v} = \mathbf{0}$.	5. A typical vector $\mathbf{v}$ in Col A has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent.
6. Given a specific vector $\mathbf{v}$, it is easy to tell if $\mathbf{v}$ is in Nul A. Just compute $A\mathbf{v}$.	6. Given a specific vector $\mathbf{v}$, it may take time to tell if $\mathbf{v}$ is in Col A. Row operations on $[\,A \quad \mathbf{v}\,]$ are required.
7. Nul $A = \{\mathbf{0}\}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.	7. Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.
8. Nul $A = \{\mathbf{0}\}$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.	8. Col $A = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$.